Application of the Delta-Rho Perturbation Method to Autonomous Orbit Computation

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Delta-Rho is a new perturbation method for the accurate numerical integration of trajectories, including those of Earth-orbiting satellites. It is most effective when used to calculate a set of dispersed orbits in the near-neighborhood of an accurately computed reference orbit. This paper considers the possible application in which the first orbit of an Earth satellite, generated in a large, ground-based computer, is taken as the reference orbit, and the second, third, and later are the dispersed orbits. Let the orbital period be T. To the extent that the sun and moon are higher harmonics of the Earth's gravitation acting on the second orbit, at time t, are the same as those which acted on the first at $(t-T)$, the method yields accurate results, with reduced computation load, relative to conventional approaches, thereby allowing for computation aboard moving vehicles with their limited computers, including the satellite itself. This paper is a report of an application of the Delta-Rho method to a geosynchronous, equatorial, circular orbit. Results of simulation studies are presented showing that with the perturbation force model truncated after the J_2 term, and five integration steps per orbit, the error after one orbit is 2.5 ft.

Introduction

THE genesis of this study was the desire to bring together a new need and a new method for determining precise position and velocity at many points in each revolution of an Earth-orbiting satellite. The need arises in missions of current interest in which the orbits of one, or perhaps many, nonmaneuvering satellites must be kept up to date in real time by computers of limited capability, that is, in computers carried aboard moving vehicles such as aircraft and the satellites themselves. The missions which require these computations frequently are such that the results must be obtained at a high level of accuracy. The accuracy requirements dictate the need for considering perturbed rather than Keplerian orbits.

In many of these applications, the orbits are approximately equal to the period of the Earth's rotation, or to some rational fraction thereof. The method of this paper is applicable to the rapid, simplified, accurate computation of such orbits.

Brief Description of Salient Features of the Delta-Rho Method

Delta-rho ($\Delta\rho$) is a new method for computing perturbed trajectories, astrodynamical as well as non-astrodynamical, to a high level of accuracy.[1] When applied to typical Earth-orbit trajectories, it becomes one of the class of methods frequently referred to as "special perturbations."

It offers the following advantages. For a given accuracy, it permits long integration step-size and a greatly simplified perturbation force model, both of which tend to reduce the computational load.

In brief, it is describable as applying Encke's method to Encke's method, which makes it, in some sense, a second-order Encke method. Although an exact method in principle, certain shortcuts are taken in this application which render it approximate rather than exact. Indeed, one of the objectives of this study was to determine whether the "shortcut $\Delta\rho$" provides sufficient accuracy for the suggested applications.

A brief summary of the $\Delta\rho$ method follows. It is applicable when the trajectory to be propagated (the "perturbed dispersed trajectory") is a near neighbor of another trajectory, called the perturbed reference trajectory, the initial time t_i on the two trajectories being the same, as shown in Fig. 1. The perturbed reference and dispersed trajectories are denoted by the six-vectors $X_0(t)$ and $X_l(t)$ respectively, in which the first three components are the Earth-centered-inertial (ECI) components of position, and the last three the components of velocity. Osculating reference and dispersed trajectories $x_0(t)$ and $x_l(t)$, shown in Fig. 1, are propagated from the known initial states $X_0(t_i)$ and $X_l(t_i)$ by analytic means. In the present study, the osculating trajectories are Keplerian, i.e., conics. (As pointed out in Ref. 1, more advanced analytically generated trajectories can also be used). It is easy to explain, in rough terms, how $\Delta\rho$ works. Because at any time t the perturbing forces are very nearly the same on the two trajectories $X_l(t)$ and $X_0(t)$, the difference $\rho_l(t)$ between the exact trajectory $X_l(t)$ and its associated osculating trajectory $x_l(t)$ is nearly the same as the difference $\rho_0(t)$ between $X_0(t)$ and $x_0(t)$. The $X_0(t)$ trajectory is computed and stored at a convenient number of time points. The $x_0(t)$ trajectory is readily generated for any time t of interest by conic propagation from the initial state. The difference vector $\rho_0(t)$ can be obtained from $[X_0(t)-x_0(t)]$. The dispersed conic $x_l(t)$ at the same time t is also readily generated by conic propagation from the known initial state. To the extent that $\rho_l(t)$ is equal to $\rho_0(t)$, the dispersed trajectory $X_l(t)$ at time t is then obtained from the dispersed trajectory osculating $x_i(t)$, and $\rho_0(t)$, by

$$X_l(t) \cong x_l(t) + \rho_0(t) \tag{1}$$

Thus, if ρ_0 captures all of the fine wiggles due to high-order perturbations present in the reference trajectory (but absent from the conic osculating trajectory x_0) then adding ρ_0 to the conic osculating trajectory x_l as is Eq. (1) will reproduce these

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wiggles into $X_I(t)$ as desired. That is, $X_I(t)$ will contain the effect of the small perturbations without the need to calculate explicitly either the perturbing accelerations or their integration to velocity and position.

The preceding description was qualitative and imprecise. To be precise, it should be recognized that the perturbing forces acting on X_I at time t are not exactly the same as those acting on X_0 at the same time, and hence $\rho_I(t)$ is not equal to $\rho_0(t)$, but rather:

$$\rho_I(t) = \rho_0(t) + \Delta\rho(t) \tag{2}$$

where $\Delta\rho(t)$ is defined to be the small correction 6-vector. Reference 1 shows that the correction vector $\Delta\rho(t)$ is obtained by numerically integrating the (exact) differential equation for $\Delta\dot{\rho}$, namely,

$$\Delta\dot{\rho}(t) = [g(X_I) - g(x_I)] - [g(X_0) - g(x_0)] + [\gamma(X_I) - \gamma(X_0)] \tag{3}$$

where $g(\cdot)$, in this application, is the central inverse-square term of the Earth's gravitational field, and $\gamma(\cdot)$ includes all of the perturbing forces to be considered, gravitational and otherwise. All terms in this equation are six-vectors. The first three elements of $\Delta\rho$ are position components, and the last three are velocity components. Hence, in $\Delta\dot{\rho}$, the first three elements are velocities, and the last three are accelerations. The first three components of $g(X_I)$ are the three velocity components of X_I, i.e. the last three components of X_I, while the last three components of $g(X_I)$ are the components of acceleration due to the Earth's inverse-square central field term acting at X_I. The complete expressions for the six-vectors are defined in Eqs. (34-49) of Ref. 1. The initial value of $\Delta\rho$, denoted by $\Delta\rho(t_i)$, is zero, in view of the definition, Eq. (2), and the initial conditions $x_0(t_i) = X_0(t_i)$ and $x_I(t_i) = X_I(t_i)$. The value of $X_I(t_i)$ is finally obtained from the correct trajectory combination equation [i.e., using Eq. (2) and discarding the approximation, Eq. (1)] in which ρ_I is added to x_I to get X_I. That is,

$$X_I(t) = x_I(t) + \rho_I(t) \tag{4}$$

$$X_I(t) = x_I(t) + \overbrace{\rho_0(t)} + \Delta\rho(t) \tag{5}$$

$$X_I(t) = x_I(t) + \overbrace{X_0(t) - x_0(t)} + \Delta\rho(t) \tag{6}$$

The last equation of the previous three is the one actually implemented, in the vehicle computer, along with the numerical integration of Eq. (3). The trajectory $X_I(t)$ obtained by this method is exact, given accurate numerical integration of the equation for $\Delta\dot{\rho}$, accounting for all known perturbing forces in γ.

The complete $\Delta\rho$ method thus consists of these steps: 1) generate a reference trajectory $X_0(t)$ from some given initial six-state $X_0(t_i)$ at initial time t_i by some conventional, highly accurate, trajectory-generating method with all desired perturbing forces included; 2) store $X_0(t)$ at a convenient number of time points $t_1, t_2, \ldots$ as will be needed by the $\Delta\dot{\rho}$ numerical integration, Eq. (3); 3) arbitrarily define the dispersed initial state $X_I(t_i)$ at the initial t_i; 4) compute $\Delta\dot{\rho}$ using Eq. (3) at each of a series of time points $t_1, t_2, \ldots$, including in $\gamma(\cdot)$ all perturbation forces of interest; 5) compute osculating trajectories $x_i(\cdot)$ and $x_0(\cdot)$, as needed, by conic propagation from the initial states $x_0(t_i)$ and $x_I(t_i)$; 6) integrate the $\Delta\dot{\rho}$ equation by some good numerical integration procedure up to time $t_1, t_2, \ldots$; and 7) at each time of interest, find $X_I(\cdot)$ by the trajectory combination equation, Eq. (6). The actual process followed is to integrate Eq. (3) over one time-step up to time t_k, and substitute the resulting $\Delta\rho(t_k)$ into Eq. (6) along with conics and stored reference trajectory data, updated to time t_k, to obtain $X_I(t_k)$. The solution $X_I(t_k)$ of Eq. (6) at each time-step is used on the right side of Eq. (3) to carry out the next step of the integration.

Relation to Previous Work

The multi-revolution method of orbit integration can produce highly accurate samples of time-varying orbital elements at one point in each revolution of a perturbed orbit, say, perigee, with considerably less computational effort than a straightforward numerical integration of the perturbed orbit.[4] The method thus appears to have some similarity to $\Delta\rho$; however, there are important differences. The method requires a capability for the accurate numerical integration through a complete revolution, and this program is used on some fraction, say ¼ to ⅓, of the total number of orbital revolutions considered. Thus, to implement it in the vehicle would require programming and storing the large, time-consuming program that we sought to remove from the vehicle computer under the objectives of the present application stated in the Introduction.

A second difference between $\Delta\rho$ and the multi-revolution method is that the latter provides one accurate point per revolution, whereas the former provides many points around the orbit, consistent with the stated objectives.

Application of $\Delta\rho$ Method to Geosynchronous Orbit

Let us now see how the $\Delta\rho$ method can be applied to calculating successive orbits of a geosynchronous satellite. We restrict our attention for present purposes to the geosynchronous satellite because its orbital period, T_0, is equal, within an acceptable tolerance, to the period of rotation of the earth. Defining the initial time to be 0, $t = T_0$ at the end of the first orbit, $t = 2T_0$ at the end of the second orbit, etc. In the treatment of perturbed orbits, the meaning of the term "period" requires elaboration, since the orbit is not strictly periodic. We shall define the orbit period T_0 as any convenient time near the end of the first orbit at which the satellite closely approaches its initial position in the ECI coordinate frame.

The first orbit must be computed by some highly accurate, conventional, orbit-computation scheme, say, in a large, capable, ground-based computer, and transferred to the vehicle. This trajectory shall then be taken as the reference orbit for $\Delta\rho$. The $\Delta\rho$ computation is then used to produce the second orbit of the satellite, based on the first as the reference, with the time points of the reference orbit all being shifted up by T_0 s, so that the initial time of the reference orbit will be equal to that of the orbit to be computed by $\Delta\rho$. Figure 2 shows the first orbit as a solid line, starting at $t = 0$ and ending at $t = T_0$. This orbit will be used as the reference orbit $X_0(t)$, except that each time point will be advanced by T_0 s. The first $\Delta\rho$ orbit starts at $t = T_0$ and ends at $t = 2T_0$. It is shown as a dashed line.

When we artificially advance all the time points along the reference trajectory by T_0, the second orbit then appears as a dispersed orbit relative to the first, starting at the same initial time T_0, in accordance with the dictates of the methodology. When $\Delta\rho$ has completed the computation of the second orbit, and is ready to start the third, the first orbit, with its time points advanced by a total of $2T_0$, can serve as the reference orbit.

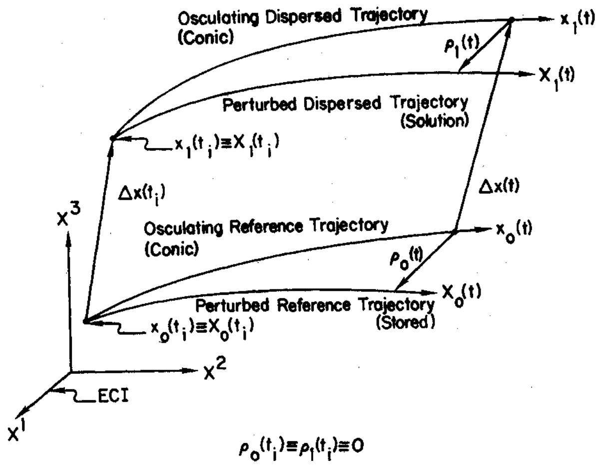
To what extent is this method of using one orbit, advanced in time, as the reference for another orbit, justified? It is accurate to the extent that the perturbing forces which acted at a point X_0 at the time the orbit was actually generated, are present again at that point in space T_0 s later. Assuming that the satellite period T_0 is equal to the Earth's rotational period, this will be valid for all of the perturbing forces generated by the non-time-varying, higher-order terms in the Earth's gravitational field. The Earth will have gone around once in the interval T_0, and therefore act at a given point in ECI at time $(t+T_0)$ exactly as it did at the same point at time t. The perturbing effects of the sun and the moon will not be the same at $(t+T_0)$ as they were at t; to the extent that they have changed, an error will be introduced. We anticipate that the lunar error will be the larger, since the moon moves further in angular position relative to ECI in T_0 than the sun does. (The moon travels through 360 deg in roughly 28 days, or about 13 deg per day. The sun travels 360 deg in roughly 365 days, or less than 1 deg per day, in this coordinate system). As shown in Ref. 3, explicit compensation can be introduced into Eq. (3) to correct for the change in perturbing effect in the one-day interval. Note that, with no compensation terms added, the basic $\Delta\rho$ method does take into account the perturbing effect of sun and moon. What is omitted is the *change* in the perturbing force in one day; the change will be a small fraction (about 1/365 for the sun, and 1/28 for the moon) of the perturbation itself.

Case Studies

This study dealt with a geosynchronous orbit, nominally equatorial and circular. The actual orbit used as the reference had the following parameters: $i=0.335981530000$ deg, $P=86157.000$ s $=23.9325$ h, and $e=2.0026942000\times10^{-5}$. The reference orbit was generated using the WGS-72 Earth model in a generalized variation-of-parameters (GVP) program.[1] The trajectory was run for a total of six orbits, all after the first being used as the "true orbit" to evaluate errors in the $\Delta\rho$ trajectories. A time-step of 29.915625000 s was used in the GVP runs, providing exactly 2880 points in one orbit. This choice of time-step was constrained by the requirements imposed by the integration rule used in the $\Delta\rho$ program. It is desirable that reference trajectory data from the GVP program be available to the $\Delta\rho$ program at any time-point selected by $\Delta\rho$'s numerical integration routine at both its major steps and at all subintervals at which it seeks to perform a function evaluation of the right-hand side of Eq. (3), so that interpolation of reference data is not required, in accordance with the strong recommendations of Ref. 2. The computation was double-precision, which provides an effective word length of about 15 decimal digits in our VAX computer. Earth gravitational harmonics out to tenth order were included, that is, up to and including $C_{10,10}$ and $S_{10,10}$. The effects of sun and moon were omitted from these runs. The integration rule was a sixth-order Runge-Kutta-Luther method, referred to in Ref. 1 as RK(6-7). We believe that the time-step and integration rule were adequate for our purposes, based on previous studies of the GVP program, although those studies had been for a different orbit and length of mission. There may be some question about the adequacy of the word length, however; the prior studies, which used a different computer, established that 64 bits or about 20 decimal digits effectively eliminated round-off error.

The start of the second orbit (the first to be generated by $\Delta\rho$) had a dispersion of 3093 ft relative to the start of the first. This is the distance between the points labeled $t=0$ and $t=T_0$ in Fig. 2, and is the initial value of the position magnitude in $[X_1(t_i)-X_0(t_i)]$ at the beginning of the $\Delta\rho$ computation. The point $X_1(t_i)$ was det rmined experimentally as a point of close approach to the initial point. (It was found later that the end of the first orbit comes within 254 ft of the starting point; the $\Delta\rho$ error in generating a trajectory from this point was unchanged).

The $\Delta\rho$ program was rewritten for this study, removing some features that had been inserted specifically for its use in Ref. 2, and adding comments to ease its interpretation by a third party. A listing of the revised program appears in Ref. 3. Written in FORTRAN77, it uses a Runge-Kutta-Sarafyan sixth-order integration rule [p. 121, (2)] requiring eight function-evaluations per major time-step. Called RK(6-8), this rule had been determined to be one of the top three in a previous study of twelve integration rules up to and including a Runge-Kutta-Shanks eighth-order method (Ref. 1, Table 4).



$\rho_0(t_i)\equiv\rho_1(t_i)\equiv0$

$\Delta\rho(t)\equiv\rho_1(t)-\rho_0(t)$

$x_k(t), X_k(t), \rho_k(t)$, k=0,1 are 6-vectors

Fig. 1 Delta-Rho vector definitions.

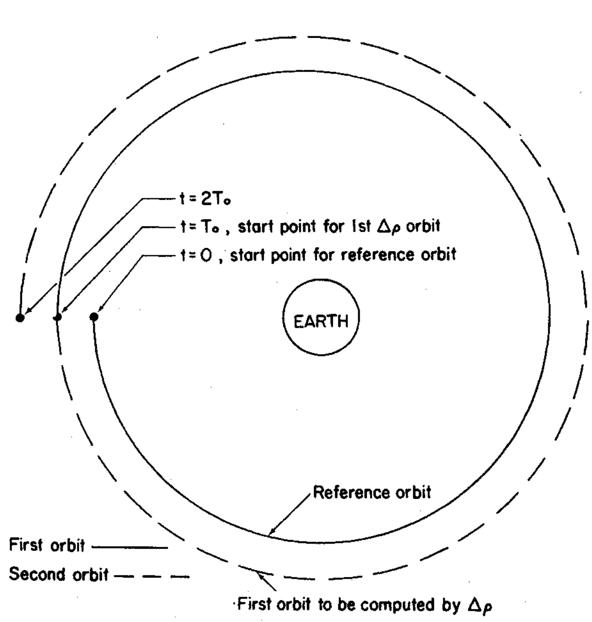


Fig. 2 Use of first orbit as the reference for the second orbit.

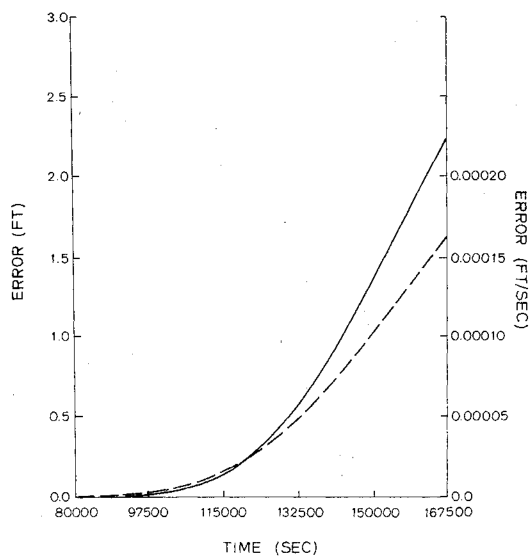


Fig. 3 Magnitude of position error (solid curve, scale at left) and velocity error (dashed curve, scale at right), first delta-rho orbit.

It has the advantage for present purposes of taking all of its data at time points equal to some integral multiple of one-eighteenth of the basic step-size. Since the smallest $\Delta\rho$ step-size we anticipated when we began the study was 538.48125 s (160 steps per orbit), compatibility dictated that the reference data from the GVP program be available at time-steps 1/18 as large, or every 29.915625 s, as noted earlier.

The perturbation force model was drastically simplified in the $\Delta\rho$ equations. It included only the J_2 term of the Earth's gravitational potential. We intentionally took this shortcut in $\Delta\rho$, to show that even with this seemingly severe reduction in fidelity, the method was still capable of producing useful results. The reason why the $\Delta\rho$ method can produce acceptable results with this reduction in the perturbation force model can be seen by referring to Eq. (3); it is the difference between the perturbation force γ at X_I, and at X_0, which determines $\Delta\dot{\rho}$. The difference between the effect of J_3, say, on X_I and its effect on X_0, at a given time, is several orders of magnitude smaller than its total effect, and therefore eliminating it entirely from the γ computation can be expected to produce negligible error. An accurate perturbation force model however must be retained in the program which generates the reference trajectory.

Results

The base case on the $\Delta\rho$ program ran the second orbit with a step-size of 2153.925 s, or 40 steps per orbit. The magnitude of the position error at the end of the first $\Delta\rho$-orbit (the end of the second GVP orbit) was 2.50065 ft, based on the GVP run at the corresponding time instant. Figure 3 shows the magnitude of the position and velocity error over the first $\Delta\rho$ orbit.

We then ran a series of runs to see if the step-size could be increased still further without increasing the final error. Table 1 presents these results. We found that the entire orbit could be traversed in one integration step if an error of 7.5 ft was acceptable at the end of one orbit. We also observed no growth of error until the step-size exceeded 17,231.4 s, corresponding to five steps per orbit. (As used here, the phrase "step-size" has the conventional meaning in numerical integration, and refers to H, the major step. The major step is broken into subintervals, which depend on the integration rule used. In our RK(6-7) integration rule, substeps occur at $H/9$, $H/6$, $H/3$, $H/2$, $2/3H$, and $5/6\ H$). The fact that the final error was insensitive to step-size over a large range suggests that the 2.5-ft error is not due to truncation, i.e., errors in the numerical integration of $\Delta\dot{\rho}$.

We ran $\Delta\rho$ for a total of five orbits, using the standard step-size. Figure 4 shows the magnitude of the position error vs. time over the five-day span. The position error magnitude reaches about 125 ft by the end of the fifth orbit. Figure 4 also shows the magnitude of the velocity error over the same interval. It remains below 0.01 ft/s.

Reference 2 found that the $\Delta\rho$ error at the end of a trajectory was very nearly proportional to the initial dispersion, other things being equal. Assuming that the points occupied by the satellite at $t = T_0, 2T_0, 3T_0, \ldots$ are roughly at a distance $d_0, 2d_0, 3d_0, \ldots$ from its position at the start of the reference orbit, then the error accumulated by $\Delta\rho$ in its second orbit will be approximately twice that accumulated in its first, etc. The total error at the end of the second $\Delta\rho$ orbit will be the sum of that accumulated in the second orbit, plus the propagation over one period of that accumulated in the first. As a result, the error growth is expected to be something larger than quadratic. The results of Fig. 4 bear this out. An error of 2.5 ft at the end of the first orbit, if it grew quadratically, would be 25 times larger or 62.5 ft after five orbits. The actual result, in which the error is 125 ft after five orbits, is therefore growing at about the expected rate. More work needs to be done on the source of the $\Delta\rho$ error and the nature of its growth with time.

Table 1 Error after first $\Delta\rho$-orbit vs step-size

Step size, s	Steps per orbit	Magnitude of position error, ft
2153.925	40	2.5
4307.85	20	2.5
8615.7	10	2.5
17,231.4	5	2.5
43,078.5	2	5.75
86,157.1	1	7.5

We then ran a study to see if rectification (reinitialization of the osculating orbits x_I and x_0 to coincide with X_I and X_0, respectively) at several points throughout the orbit could decrease the error. If the error was primarily due to truncation error in the numerical integration of $\Delta\dot{\rho}$, then rectification should reduce the error, as it would in a standard Encke orbit computation. However, we found that when we rectified every 10 steps around the orbit, the error at the end of the first $\Delta\rho$ orbit was virtually unchanged. This adds further support to the suggestion that errors in the numerical integration of $\Delta\dot{\rho}$ are not the primary cause of the observed error.

A test in which the J_3 term was included in the $\Delta\rho$ perturbation-force model showed no change in the error at the end of one orbit, confirming the claim the $\Delta\rho$ can operate successfully with a highly simplified perturbation force model.

Figures 5-7 are presented as an aid to understanding how the $\Delta\rho$ method functions. Figure 5 shows the magnitude of the

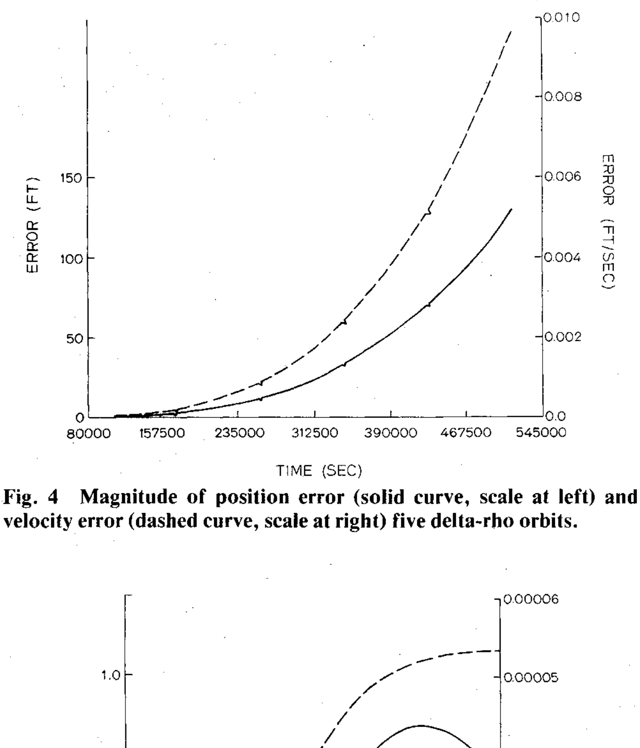


Fig. 4 Magnitude of position error (solid curve, scale at left) and velocity error (dashed curve, scale at right) five delta-rho orbits.

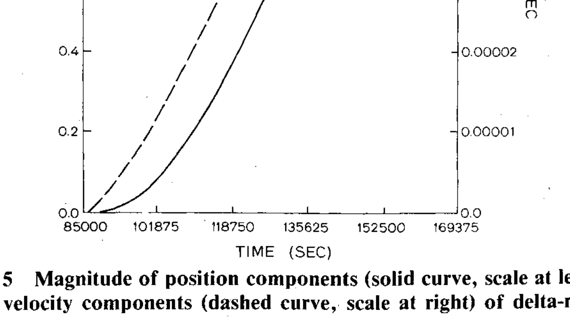


Fig. 5 Magnitude of position components (solid curve, scale at left) and velocity components (dashed curve, scale at right) of delta-rho vector.

three position-components and three velocity-components of $\Delta\rho$ for the base case. Figure 6 is the magnitude of the first and second three components of $\Delta\dot{\rho}$. (Obviously, the magnitude of the first three components of $\Delta\dot{\rho}$ is the same as that of the second three components of $\Delta\rho$ —see dashed curves of Figs. 5 and 6.) In Eq. (3) for $\Delta\dot{\rho}$, it is of interest to know whether the central-field terms $\{[g(X_I)-g(x_I)]-[g(X_0)-g(x_0)]\}$, or the perturbation terms $[\gamma(X_I)-\gamma(X_0)]$, dominate. (Rectification would normally be provided whenever the former exceeds the latter by some specified amount, since rectification resets the former to zero.) Figure 7 provides this information. The solid curve is the magnitude of the acceleration components of the first of the two previous collections of terms, and the dashed curve is the magnitude of the second. The $\gamma(\cdot)$ terms, which represent the difference between the effect of J_2 on the reference trajectory and that on the dispersed trajectory, are seen to range between 0.3×10^{-9} and 0.6×10^{-9} ft per second squared (fpss), while the central-field contributions to the acceleration inputs into $\Delta\dot{\rho}$ range from 0.06 to 0.4×10^{-8} fpss. The magnitude of the vector sum of these acceleration inputs appears as the solid curve in Fig. 6. This level of acceleration is that which ultimately gets integrated to the $\Delta\rho$ position-magnitude in Fig. 5 (solid curve), which is seen to be below one ft throughout the whole trajectory. Reviewing the curves of acceleration, velocity, and position, or more specifically, their component-by-component counterparts, shows that they vary smoothly, and hence their numerical integration does not pose a difficult computational problem.

It is instructive to notice that the magnitude of the position-components of the difference ρ_0 between the reference trajectory X_0 and its osculating companion x_0 exceeds 60,000 ft at the end of one orbit (Fig. 8). This is the difference vector that would be obtained if one used Encke's method in its classical form to generate the reference trajectory. The 60,000 ft represents the total integrated effect of the perturbing forces in the GVP program over one orbit. The higher-order nature of the $\Delta\rho$ method compared to Encke's method becomes apparent when one compares ρ_0 at 60,000 ft in Encke's method to $\Delta\rho$ at less than one ft over the same time interval. Since $\Delta\rho$ represents the difference between the integrated effect of the perturbing accelerations on the reference trajectory and that on the dispersed trajectory, it can be said that the perturbing forces, to within 1 part in 60,000, do have very nearly the same effect on the dispersed and reference trajectories, as claimed.

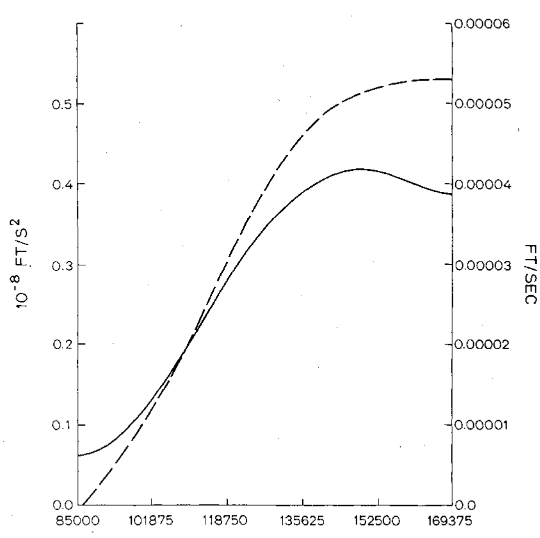


Fig. 6 Magnitude of first three components (dashed curve, scale at right) and last three components (solid curve, scale at left) of delta-rho-dot vector.

The fact that the integrated magnitude of $\Delta\rho$ never exceeds one ft, which is less than 2.5 ft position error at the end of the first orbit, suggests once more that numerical integration of $\Delta\rho$ is not the primary source of the 2.5 ft error.

As yet we have not isolated the cause of this error. The suspicion that it may be the result of the conic propagation, in which Kepler's equation is solved iteratively until a convergence criterion is met, was dismissed when decreasing the convergence criterion by two orders of magnitude made essentially no change in the trajectory error at the end of the first orbit. We now question whether the error is due to $\Delta\rho$, or to the GVP reference program generator. A further test of the GVP program in which the step-size in that program was reduced by a factor of three shows no change in the GVP trajectory, hence no change in the error, and thus indicates that truncation error in GVP does not appear to be significant. This leaves round-off error in GVP as a possible cause.

Implementation Considerations

We have suggested the possibility that $\Delta\rho$ may be a useful approach to certain satellite problems. We have not touched on the details of how, in respect to hardware, the reference data is to be transferred from the ground to the vehicle. Depending on the accuracy requirements of a given mission, and the length of the mission, it may be necessary, in order to bound the error-growth of $\Delta\rho$, that the reference trajectory be updated from time to time by the ground. Alternatively, several reference trajectories, spanning the future mission interval, could be furnished from the ground at the initial time, but this obviously adds to the storage requirements aboard the vehicle.

What must be stored for one point on the reference orbit is

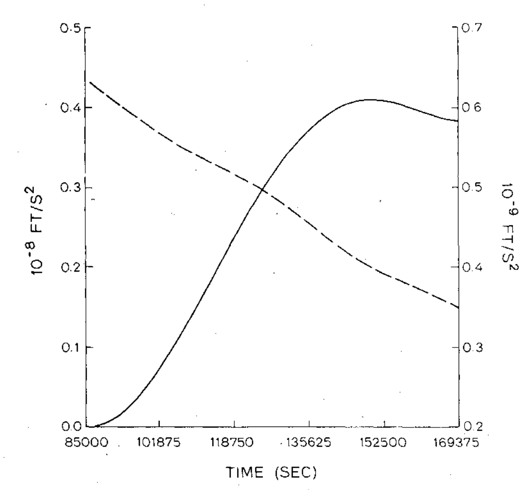


Fig. 7 Magnitude of acceleration components of $[g(X_I)-g(x_I)]-[g(X_0)-g(x_0)]$ (solid curve, scale at left) and $[\gamma(X_I)-\gamma(X_0)]$ (dashed curve, scale at right).

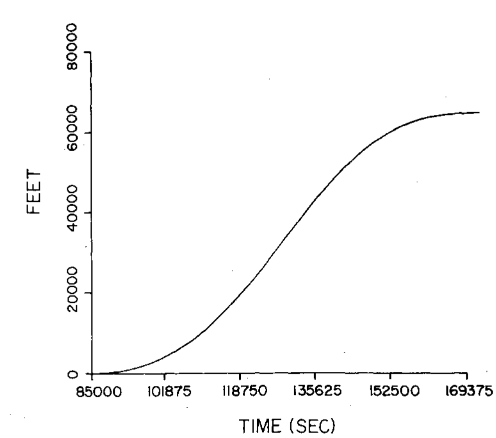


Fig. 8 Magnitude of position components of ρ_0.

the seven-vector consisting of the six-vector $X_0(t_k)$, and the associated time t_k. If one were to use the same RK(6-8) integration rule as used herein, and designed $\Delta\rho$ for five major step around the orbit, then X_0 would have to be stored at a minimum of $7\times5+1=36$ time points, since each Runge-Kutta integration-step uses data at seven time points, in addition to the initial time. Correspondingly more storage would be required if one wanted to produce $\Delta\rho$ output at more than five points around the orbit.

If five $\Delta\rho$ steps per revolution are adequate for a given mission, and if the RK(6-8) integration is used for $\Delta\rho$, then the step-size of the reference program generator could be increased to approximately 957 s, to be consistent with the constraint of providing data at every major step and subinterval of the RK(6-8) integration. (This assumes that the larger step-size maintains the accuracy of the reference orbit.)

The need for the ground to update the reference orbit stored in the vehicle from time to time is not only a burden, but also an opportunity, since, in the case of a satellite whose orbit changes in ways beyond the capability of orbit models to predict (say, for example, due to atmospheric drag at perigee in a highly eccentric orbit), the ground may well be updating its own ephemeris continuously with measurements from tracking stations. Updating the reference orbit could be quite consistent with operational considerations in such a situation.

Further Extensions

The shortcut $\Delta\rho$-method suggested here for synchronous orbits is easily extended to the case in which the orbital period T_0 is equal to a rational fraction m/n of the Earth's rotational period T_E. In such a case, one would take an interval of length mT_E for the reference orbit, and would apply this to each group of n revolutions to be integrated by $\Delta\rho$. For example, in the Global Positioning System (GPS), where the period is 1/2 T_E, 24 h of data would be used as the reference for successive pairs of $\Delta\rho$-orbits.

Figure 5 shows that the magnitude of the three position-components of the $\Delta\rho$ vector is less than 1 ft in the orbit under study. This being true, one could further simplify the $\Delta\rho$ equations in this application, making a further substantial decrease in the length of the code required, by omitting entirely the differential equation for $\Delta\dot{\rho}$, that is, by setting $\Delta\rho$ itself equal to the 0-vector, from which it follows that $\rho_I(t)$ is equal to $\rho_0(t)$ at all values of time. The totality of the method then consists, as suggested in the discussion surrounding Eq. (1), of adding $\rho_0(t)$ to $x_I(t)$. To implement this method, one needs only a conic propagation routine to develop $x_I(t)$ from initial condition $X_I(t_i)$, and storage of the vector $\rho_0(t)$ for one orbit, plus logic for correcting time modulo T_0. The accurate ground-based computation which produced $X_0(t)$ could easily produce $x_0(t)$ and hence $\rho_0(t)=X_0(t)-x_0(t)$. Then $\rho_0(t)$ would be stored in the vehicle-borne computer, rather than $X_0(t)$. The total error of such a scheme should be less than 3.5 ft for one orbit of the type considered—the sum of the error observed on $\Delta\rho$, plus the maximum value of $\Delta\rho$ itself. Further work needs to be done to examine the limits of this concept.

When requirements are such that setting $\Delta\rho(t)$ equal to zero yields results of acceptable accuracy, then another possible way of simplifying the computation in the vehicle would be to combine the $\Delta\rho$-concept with the multi-revolution method of orbit integration. In this approach, the ground sends to the vehicle, once each revolution, a six-vector of orbital elements, and time, which are used to define the initial condition $X_I(t_i)$ of the perturbed dispersed trajectory of Fig. 1, for the next revolution. The vehicle computer need perform only these minimal computations: 1) store $\rho_0(t)$ at a convenient number of time points $5_i,\ldots t_k,\ t_{k+1},\ldots t_i+T_0$ around one revolution, this data also having been generated on the ground and sent to the vehicle at an earlier time, 2) accept the initial-condition vector $X_I(t_i)$ from the ground once per revolution, 3) generate the osculating dispersed trajectory $x_I(t)$ by conic propagation from the initial state $X_I(t_i)$ to time points $t_k,\ t_{k+1},\ldots$, and 4) add the vector $\rho_0(t_k)$ to the vector $x_I(t_k)$ as in Eq. (1) to get $X_I(t_k)$, the desired result. The computer would also have to update the reference orbit by T_0, $2T_0,\ldots$ at successive revolutions. The total package should lead to a very modest-size program for the vehicle computer.

Conclusions

The results presented illustrate that the Delta-Rho ($\Delta\rho$) method can be applied to orbits whose period is essentially that of Earth's rotation by incrementing time at points on the reference orbit by multiples of the orbital period. The method is thus easily applied to any geosynchronous orbit. It can also be applied to orbits which are rational fractions of the Earth's rotational period, such as the 12-h Global Positioning System orbit, by using 24 h of data as the "reference orbit."

These results illustrate that truncating the Earth's gravitational field after the J_2 term in the position difference ($\Delta\rho$) equations leads to satisfactory results (if a 2.5-ft error is satisfactory) compared to the tenth-order model implemented in the generalized variation of parameters program. This provides a major reduction in computer code required. This is not to say that the gravitational harmonics above J_2 are unimportant; they must be included in the reference trajectory if the accuracy is to be maintained. However, once they are properly included in the reference trajectory, then the $\Delta\rho$-computed dispersed trajectories will have these effects built into them, without their being explicitly calculated in the differential equations for the position difference $\Delta\rho$. This is one of the remarkable attributes of the $\Delta\rho$ method.

The results show that, for the circular, equatorial, geosynchronous orbit, an integration step-size of one-fifth the orbital period produces no observable truncation error. The $\Delta\rho$ step-size can be $18j$ times larger than that used in the reference-trajectory generator, where j is a positive integer. Thus, a big time saving can be realized in a vehicle-borne computer in which the $\Delta\rho$ method is implemented.

Rectification four times per orbit produced no appreciable improvement in accuracy in the orbit studied. This is probably because the position difference $\Delta\rho$ itself, at less than 1 ft, was smaller than the observed error of 2.5 ft at the end of one orbit, and hence, expected improvements in the numerical integration of $\Delta\dot{\rho}$ are masked by the other sources of error.

The results show that error is about 2.5 ft after one orbit and about 125 ft after the fifth orbit generated by the $\Delta\rho$ calculations. These results were obtained without the effects of sun and moon in the reference trajectory. Had the sun and moon been included in the reference trajectory, the errors could be expected to be larger. However, for the same reason that J_2 suffices in $\Delta\rho$ while harmonics out to tenth order are used in the reference, it is expected that the additional error incurred in the $\Delta\rho$ orbit when the sun and moon are properly included in the reference trajectory will be much smaller than the solar/lunar perturbation of the reference trajectory itself.

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